

Pseudoriemannian symmetric spaces: one-type realizations and open embeddings to grassmannians

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Consider a real semisimple group G . Let σ be an automorphism of G of order 2 (i.e. $\sigma^2 = 1$). Denote by H the set of fixed points of the automorphism σ . Homogeneous spaces G/H are known as pseudoriemannian *symmetric spaces* (another term – affine symmetric spaces; below we name them by symmetric spaces). According to Berge classification [1] there exists 54 series of classical pseudoriemannian symmetric spaces and 131 exceptional spaces ². This list contains many interesting objects from various branches of mathematics. In particular it contains riemann symmetric spaces (including spheres, Lobachevskii spaces, Siegel upper half plane, grassmannians, matrix balls, Cartan domains, future tubes, quadrics in \mathbb{CP}^n , symmetric cones, moduli space of $K3$ -surfaces), complex symmetric spaces (including the space $\mathrm{PGL}(n, \mathbb{C})/\mathrm{O}(n, \mathbb{C})$ of all nondegenerate quadrics), simple Lie groups³, multidimensional hyperboloids, spaces of correlations.

Harmonic analysis for many symmetric spaces appeared long ago. Analysis on sphere and Lobachevskii plane is a classical subject (from A.M.Legendre to F.G.Mehler, P.Funk, J.Radon). Analysis on semisimple groups was a subject of investigations of I.M.Gelfand and M.A.Naimark, Harish-Chandra and their numerous successors. Analysis on riemann symmetric spaces was a subject of works of Hua Loo Keng, S.G.Gindikin and F.I.Karpelevich, S.Helgason. Analysis on hyperboloids was investigated by V.F.Molchanov. Studying of analysis on arbitrary pseudoriemannian symmetric spaces was initiated by M.Flensted-Jensen work about discrete series (H.Schlichtkrull, T.Oshima, J.Sekiguchi, V.F.Molchanov, E.P. van den Bahn, P.Delorm). Now this subject is one of the most wide and interesting branches of noncommutative harmonic analysis.

The purpose of this paper is to formulate two simple observations (by strange way they were not known ⁴). The first observation is:

all 54 series of classical pseudoriemannian symmetric spaces have extremely simple uniform geometric realizations. Precisely a point of a symmetric space is

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²Of course classification is given up to coverings and connected components. A symmetric space G/H is called classical if the group G is classical (in this case the group H also is classical). Number of series in various versions of the list is different. The reason is preasence of the spaces $\mathrm{O}(n+2, \mathbb{C})/\mathrm{O}(n, \mathbb{C}) \times \mathrm{O}(2, \mathbb{C})$ (and their real forms). Properties of these spaces are very specific and hence these spaces often are not included to the series $\mathrm{O}(n+m, \mathbb{C})/\mathrm{O}(n, \mathbb{C}) \times \mathrm{O}(m, \mathbb{C})$.

³Let P be a simple Lie group. Then the group $G = P \times P$ acts on P by left and right multiplications, the stabilizer of the point e is $H = P$

⁴I think the reason is the length of the Berge list and nonroot nature of our construction

a pair of complementary subspaces V_1, V_2 in $\mathbb{R}^k, \mathbb{C}^k, \mathbb{H}^k$ satisfying very simple conditions (isotropy, orthogonality or existence of fixed operator transposing V_1 V_2).

It seems to me that this observation pleasantly simplifies the picture. It also gives a possibility to work with arbitrary *classical* symmetric spaces by uniform elementary methods. Several applications of this point of view are in section §3. In particular for all classical symmetric spaces we introduce simple matrix coordinates (this construction generalizes Cartan matrix balls [3]).

The second observation is:

all classical symmetric spaces have natural open embeddings to grassmannians.

This fact was known for many particular cases. It was many discussed from geometrical point of view and it was intensively used in harmonic analysis (see for instance the work of Hua [7], more general observations see in papers of B.O.Makarevich[8], W.Bertram [2] and S.G.Gindikin [6], the last paper also contains some bibliography; I have to apologize to all authors which are not cited here). We show that this embedding exists for all⁵ classical symmetric spaces. Moreover in our model it is absolutely obvious. Direct consequence of this observation is the following fact (see precise formulation in Section 3):

Representation of the group G in $L^2(G/H)$ is the restriction of some representation of some larger group $G^ \supset G$.*

§0. Notations

Here for avoiding of ambiguity we fix a terminology and notations.

0.0. The symbol \mathbb{K} denote \mathbb{R}, \mathbb{C} or quaternionic ring \mathbb{H} . The term *a linear space over \mathbb{H}* means (for us) a right module over \mathbb{H} . It is convenient to think that elements of \mathbb{H}^n are vector-columns v . We wright linear operators over \mathbb{H} in a form $v \mapsto Av$ where A is a matrix. We wright a multiplication by a scalar in a form $v \mapsto v\lambda$.

An *antilinear operator* in a linear space V over \mathbb{C} is a map $V \mapsto V$ satisfying the conditions

$$A(v + w) = Av + Aw \quad A\lambda v = \bar{\lambda}Av, \quad \lambda \in \mathbb{C}$$

REMARK. Antilinear maps in linear spaces over \mathbb{H} don't exist. Indeed let A be an antilinear map. Then

$$\begin{aligned} A(v\lambda\mu) &= A((v\lambda)\mu) = (A(v\lambda))\bar{\mu} = (Av)\bar{\lambda}\bar{\mu}, \\ A(v\lambda\mu) &= A(v(\lambda\mu)) = (Av)\overline{\lambda\mu} = (Av)\bar{\mu}\bar{\lambda} \end{aligned}$$

⁵up to covering, center and components of connectedness

0.1. Forms. We remind that a map $B(v, w)$ from $\mathbb{K}^n \times \mathbb{K}^n$ to \mathbb{K} is named *sesquilinear* if for all $v, v_1, v_2, w, w_1, w_2 \in \mathbb{K}^n$, $\lambda \in \mathbb{K}$ we have

$$\begin{aligned} B(v\lambda, w) &= B(v, w)\lambda & B(v_1 + v_2, w) &= B(v_1, w) + B(v_2, w) \\ B(v, w\lambda) &= \bar{\lambda}B(v, w) & B(v, w_1 + w_2) &= B(v, w_1) + B(v, w_2) \end{aligned}$$

A sesquilinear map is a *hermitian form*, if $B(v, w) = \overline{B(w, v)}$ $v, w \in \mathbb{K}^n$. A sesquilinear map is a *antihermitian form* if $B(v, w) = -\overline{B(w, v)}$.

The term *form* everywhere in this paper means a *nondegenerate* form on a linear space over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ having one of the following 7 types:

- over \mathbb{R} – bilinear symmetric and skew symmetric forms
- over \mathbb{C} – bilinear symmetric and antisymmetric forms and also hermitian and antihermitian forms
- over \mathbb{H} – hermitian and antihermitian forms

REMARKS. a) Of course hermitian (antihermitian) forms over \mathbb{R} are the same as symmetric (skew symmetric) forms.

b) Antihermitian forms over \mathbb{C} differ unessentially from hermitian forms. Indeed let $B(v, w)$ be a antihermitian form. Then the form $iB(v, w)$ is hermitian.

c) Bilinear forms on \mathbb{H}^n don't exist (an expression $\sum x_s y_s$ is not bilinear form on right \mathbb{H} -module!).

d) Hermitian (antihermitian) forms over \mathbb{H} can be represented as $B(v, w) = \sum \bar{w}_s b_{st} v_t$ where $b_{st} = \bar{b}_{ts}$ (resp. $b_{st} = -\bar{b}_{ts}$).

Remind the classification of forms up to a linear change of variables. Hermitian forms⁶ over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are enumerated by inertia indexes. In all other cases all nondegenerate forms of the given type on a given linear space are equivalent.

0.2. Classical groups. We denote by $\mathcal{U}(B)$ the group of all linear operators preserving a form B . Fix notations for all 7 types of the groups $\mathcal{U}(B)$:

$\text{Sp}(2n, \mathbb{R}), \text{Sp}(2n, \mathbb{C})$ – the groups of all linear operators in $\mathbb{R}^{2n}, \mathbb{C}^{2n}$ preserving skewsymmetric bilinear form.

$\text{O}(n, \mathbb{C})$ – the group of operators in \mathbb{C}^n preserving symmetric bilinear form.

$\text{O}(p, q), \text{U}(p, q), \text{Sp}(p, q)$ – the groups of operators in $\mathbb{R}^{p+q}, \mathbb{C}^{p+q}, \mathbb{H}^{p+q}$ preserving a hermitian form with inertia indexes (p, q)

$\text{SO}^*(2n)$ – the group of operators in \mathbb{H}^n , preserving antihermitian form.

By the term "*classical group*" we mean a group of this 7 series and also

$\text{GL}(n, \mathbb{R}), \text{GL}(n, \mathbb{C}), \text{GL}(n, \mathbb{H})$ – groups of all linear operators in $V = \mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$.

We also use notation $\text{GL}(V)$.

Emphasis that *we don't include to this list groups*

$$\text{SL}(n, \mathbb{R}), \text{SL}(n, \mathbb{C}), \text{SL}(n, \mathbb{H}), \text{SU}(p, q), \text{PSL}(n, \mathbb{R}), \text{SL}^\pm(n, \mathbb{R}) \text{ etc.}$$

⁶including antihermitian forms over \mathbb{C}

Accordingly we consider (for instance) symmetric spaces $U(p, q)/O(p, q)$ and don't consider $SU(p, q)/SO(p, q)$. From the point of view of harmonic analysis there is no difference between spaces $U(p, q)/O(p, q)$ and $SU(p, q)/SO(p, q)$. From the point of view of our realization first space is more pleasant.

0.3. Grassmannians. Denote by $\text{Gr}_p(V)$ the set of all p -dimensional subspaces in a linear space V .

Consider a form B in V . Remind that a subspace $Q \subset V$ is named *isotropic* with respect to the form B , if B equals identical zero on Q .

Recall that a form B in V is named *split*, if there exists B -isotropic half-dimensional subspace. Recall that a hermitian form is split if its positive and negative inertia indexes coincides. Orthogonal(symmetric bilinear) forms over \mathbb{C} and antihermitian forms over \mathbb{H} are split iff dimension of the space is even. Skew-symmetric forms always are split.

If B is split then there exist a basis $e_1, \dots, e_n, f_1, \dots, f_n$ in the space V such that

$$B(e_k, e_l) = 0, \quad B(f_k, f_l) = 0, \quad B(e_k, f_l) = \delta_{k,l}$$

Let B be a *split* form. Denote by $\text{Gr}(V, B)$ a set of all maximal isotropic subspaces in V .

A word *grassmannian* below means $\text{Gr}_p(V)$ or $\text{Gr}(V, B)$.

§1. Semiinvolutions and their centralizer

1.1. Semiinvolutions. A *semiinvolution* (for detailed discussion and definition for arbitrary division rings see Dieudonne book [4]) in a linear space V over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is a linear or antilinear operator J satisfying the condition

$$J^2 = \lambda$$

where λ is an element of center of \mathbb{K} .

LEMMA 1.1. *Let J be an antilinear semiinvolution over \mathbb{C} , let $J^2 = \lambda$. Then $\lambda \in \mathbb{R}$.*

PROOF. Calculate J^3 by two ways:

$$J^3 v = J J^2 v = J \lambda v = \overline{\lambda} v \quad J^3 v = J^2 J v = \lambda J v \quad \blacksquare$$

Consider a map $S(J) : \text{Gr}_p(V) \rightarrow \text{Gr}_p(V)$ given by the formula $Q \mapsto JQ$. Obviously $S(J)^2 = 1$. The main object of our interest are these maps. For each element σ of center of \mathbb{K} we have $S(\sigma J) = S(J)$. By this reason we assume

$$\boxed{J^2 = \pm 1}$$

Moreover if $\mathbb{K} = \mathbb{C}$ and J is linear then we can assume $J^2 = 1$.

Denote by $\text{GL}^J = \text{GL}^J(V)$ the centralizer of the semiinvolution J in the group $\text{GL}(V)$.

1.2. Description of semiinvolutions. A semiinvolution J defines an additional structure in a linear space V . This structure is discussed below.

a) Let J be linear, and $J^2 = 1$. Consider the subspaces $V_{\pm} \subset V$ which consist of vectors v satisfying conditions $Jv = \pm v$. Then $V = V_+ \oplus V_-$. Thus the semiinvolution J defines the fixed decomposition of V into a direct sum of two subspaces. Obviously $\text{GL}^J = \text{GL}(V_+) \times \text{GL}(V_-)$.

b) Let $\mathbb{K} = \mathbb{R}$, $J^2 = -1$, $V = \mathbb{R}^{2n}$. Then we can consider the space V as a space over field \mathbb{C} where multiplication by a scalar i is the operator J . Obviously $\text{GL}^J \simeq \text{GL}(n, \mathbb{C})$.

c) Let $\mathbb{K} = \mathbb{C}$, J is antilinear, $J^2 = -1$. Define the action of the algebra \mathbb{H} on V . For these purpose we assume that the subalgebra $\mathbb{C} \subset \mathbb{H}$ acts as it acts and that the quaternionic imaginary unit j is the operator J .

Thus we define a structure of a linear space over \mathbb{H} on the space V . Clearly the group GL^J is a quaternionic group GL .

d) Let $\mathbb{K} = \mathbb{C}$, $\dim V = k$, let J be antilinear, $J^2 = 1$. Consider the set W of all fixed points of the involution J . Obviously the set W is a \mathbb{R} -subspace in V and $V = W \oplus iW$. Thus we can consider V as a complexification of the real linear space W . Obviously $\text{GL}^J \simeq \text{GL}(k, \mathbb{R})$.

e) Let $\mathbb{K} = \mathbb{H}$, $\dim V = k$, $J^2 = -1$. Consider the set W of all points $v \in V$ satisfying the condition

$$Jv = vi$$

It is clear that W is a linear space over \mathbb{C} and $V = W \oplus Wj$. Hence the space V is a "quaternionization" of k -dimensional complex space W (i.e. $V = W \otimes_{\mathbb{H}} \mathbb{H}$). Obviously $\text{GL}^J \simeq \text{GL}(k, \mathbb{C})$.

We say that semiinvolution is *split* if there exists a subspace Q such that $V = Q \oplus JQ$. Only this case is interesting for us. All semiinvolutions of types b)–e) always are split. An involution of the type a) is split iff $\dim V_+ = \dim V_-$.

1.3. Semiinvolutions consistent with forms. Let B be a form, let J be a *linear* semiinvolution. We say that the semiinvolution J is consistent with the form B if for all v, w we have

$$B(Jv, Jw) = \mu B(v, w) \tag{1.1}$$

where μ is an element of the center of the division ring \mathbb{K} .

We say that an *antilinear* semiinvolution J is consistent with a form B if the following condition is fulfilled

$$B(Jv, Jw) = \mu \overline{B(v, w)} \tag{1.2}$$

Now we will show that $\mu = \pm 1$

LEMMA 1.2. a) Let J be linear. Then $\mu = \pm 1$.

b) Let J be antilinear and the form B is hermitian. Then $\mu = \pm 1$.

c) Let J be antilinear and the form B is bilinear. Then there exists a constant σ such that for the form σB we have $\mu = 1$.

PROOF. a) Bearing in mind the condition $J^2 = \pm 1$ we obtain

$$B(v, w) = B(J^2 v, J^2 w) = \mu B(Jv, Jw) = \mu^2 B(v, w)$$

$$\text{b) } B(v, w) = B(J^2 v, J^2 w) = \mu \overline{B(Jv, Jw)} = \mu \bar{\mu} B(v, w)$$

Hence $|\mu| = 1$. Further we substitute $v = w$ to (1.2) and observe that $B(v, v)$ is real.

c) Similarly we obtain $|\mu| = 1$. Assume $C(v, w) = \frac{1}{\sigma} B(v, w)$. Then

$$\sigma C(v, w) = \mu \overline{\sigma C(v, w)}$$

and now the statement becomes obvious. ■

LEMMA 1.3. *Let J be a semiinvolution consistent with the form B . Let Q be a subspace isotropic with respect to the form B . Then the subspace JQ is isotropic with respect to B .*

PROOF. If $B(v, w) = 0$ then $B(Jv, Jw) = 0$. ■

Thus the semiinvolution J defines an involution on B -isotropic grassmannian.

1.4. Managing form. Fix a form B and a semiinvolution which is consistent with J . Consider the expression

$$D(v, w) := B(v, Jw)$$

It is easy to check (in each particular case it is obvious) that $D(v, w)$ is a "form" in our sense.

EXAMPLE. Let B be a symmetric bilinear form over \mathbb{R} . Fix λ and μ . Then

$$D(v, w) = B(v, Jw) = \mu B(Jv, J^2 w) = \mu \lambda B(Jv, w) = \mu \lambda B(w, Jv) = \mu \lambda D(w, v)$$

and the type of the form D becomes obvious.

LEMMA 1.4. *Let a subspace Q be isotropic with respect to the form B . Then the subspace JQ is orthogonal Q with respect to managing form D .*

PROOF. Let $v, w \in Q$. Then $D(v, Jw) = B(v, J^2 w) = 0$. ■

Hence we have 3 structures in the space V : two forms and the semiinvolution. If we know two of these structures we can reconstruct the third structure. We can formulate this remark in the following form:

Denote by $\mathcal{U}(B)^J$ the set of all elements of $\mathcal{U}(B)$ commuting with J . Then

$$\mathcal{U}(B)^J = \mathcal{U}(B) \cap \text{GL}(V)^J \tag{1.3}$$

$$\mathcal{U}(B)^J = \mathcal{U}(B) \cap \mathcal{U}(D) \tag{1.4}$$

$$\mathcal{U}(B)^J = \mathcal{U}(D) \cap \text{GL}(V)^J \tag{1.5}$$

1.5. Split pairs (B, J) . Let J be a semiinvolution which is consistent with a form B . We say that the pair (B, J) is *split*, if there exists an isotropic

subspace Q in V such that $V = Q \oplus JQ$. We also say that V is a *spiting* subspace.

LEMMA 1.5. *Let (B, J) be a split pair. Then for each maximal isotropic subspace P in V the subspace JP coincides with D -orthogonal complement to P .*

PROOF. The statement follows from Lemma 1.4 and calculation of dimensions. ■

PROPOSITION 1.6. *Fix a type of a form B , the type of a semiinvolution J and number $\mu = \pm 1$. Then there exists an unique (up to linear change of coordinates) a split pair (B, J) of the given type.*

PROOF. It is easy to see that the pair (B, J) is uniquely defined by the restriction of managing form D to the subspace Q . Now the problem is reduced to classification of forms. ■

1.6. Description of groups $\mathcal{U}^J(B)$ for split pairs (B, J) . In this subsection we give long enumeration of species parallel to enumeration of Subsection 1.2. We preserve numeration a), b), c) etc. from Subsection 1.2.

In fact it is possible to carry out the same work in general form for arbitrary division rings and not only for split pairs (B, J) , see. ([4]).

a) Let J be linear, $J^2 = 1$. Obviously subspaces V_{\pm} are $\mathcal{U}^J(B)$ -invariant. Hence elements of the group $\mathcal{U}^J(B)$ have block structure

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \quad (1.6)$$

We consider two subcases.

a1) Let $\mu = +1$. Then subspaces V_+ , V_- are orthogonal with respect to the form B . Denote by B_{\pm} the restriction of the form B to V_{\pm} . Obviously $\mathcal{U}^J(B) = \mathcal{U}(B_+) \times \mathcal{U}(B_-)$.

A splitting subspace P is a graph of invertible operator $L_P : V_+ \rightarrow V_-$. Clearly operator L_P identifies the form B_+ with the form $(-B_-)$ (it is equivalent to isotropy of subspace P). Hence $\mathcal{U}(B_+) \simeq \mathcal{U}(B_-)$.

a2) Assume $\mu = -1$. Let $v_1, v_2 \in V_+$. Then

$$B(v_1, v_2) = -B(Jv_1, Jv_2) = -B(v_1, v_2)$$

Hence the subspaces V_{\pm} are B -isotropic. The form B defines nondegenerate pairing between V_+ and V_- . Hence the operator g_1 (see (1.6)) is contragredient to g_2 with respect to our pairing.

Thus $\mathcal{U}^J(B) = \text{GL}(V_+)$

b) Let $\mathbb{K} = \mathbb{R}$, $J^2 = -1$. Then our space over \mathbb{R} can be considered as a space over \mathbb{C} . Let us define in this space a \mathbb{C} -valued form

$$Z(v, w) = B(v, w) + iD(v, w) = B(v, w) + iB(v, Jw)$$

Now $\mathcal{U}^J(B) = \mathcal{U}(Z)$.

c) Let $\mathbb{K} = \mathbb{C}$, let J be antilinear, $J^2 = -1$. Let us define in our \mathbb{H} -space a \mathbb{H} -valued form

$$Y(v, w) = B(v, w) + jD(v, w) = B(v, w) + jB(v, Jw)$$

We have $\mathcal{U}^J(B) = \mathcal{U}(Y)$.

d) Let $\mathbb{K} = \mathbb{C}$, let J be antilinear, $J^2 = 1$. Consider the \mathbb{R} -subspace $V_{\mathbb{R}}$ consisting of fixed points of the semiinvolution J . Consider the restriction of the form B to $V_{\mathbb{R}}$. For all $v, w \in V_{\mathbb{R}}$ we have

$$B(v, w) = \overline{\mu B(Jv, Jw)} = \overline{\mu B(v, w)}$$

Now define the form X on $V_{\mathbb{R}}$ which equals B if $\mu = 1$, and equals iB if $\mu = -1$. Then the form X is a \mathbb{R} -value form and $\mathcal{U}^J(B) = \mathcal{U}(X)$.

e) Let $\mathbb{K} = \mathbb{H}$, $J^2 = -1$. This case is similar to d).

§2. Realizations of classical symmetric spaces

In fact models of all classical symmetric spaces were obtained in previous section. We have only to write the list.

2.1. List 1. The case of split pairs (B, J) . Fix a space $V = \mathbb{K}^\alpha$ and split pair (B, J) in this space (see 1.5). We will say that the form B is an *underlying form*, and a semiinvolution J is a managing *semiinvolution*. Let Q be the associated managing form, see 1.4.

Let us define the space $\mathcal{S}(B, J)$. Its points are ordered pairs of subspaces Q_1, Q_2 in V satisfying the conditions

1. Q_1, Q_2 are maximal B -isotropic subspaces
2. $V = Q_1 \oplus Q_2$.
3. $JQ_1 = Q_2$ or equivalently $JQ_2 = Q_1$ (or equivalently Q_2 is D -orthogonal complement to Q_1)

Consider the centralizer $G = \mathcal{U}^J(B)$ of the semiinvolution J in the group $\mathcal{U}(B)$. The group $G = \mathcal{U}^J(B)$ acts on the space $\mathcal{S}(B, J)$ by the obvious way. We claim that either $\mathcal{S}(B, J)$ is a symmetric space G/H or $\mathcal{S}(B, J)$ is an union of finite family of symmetric spaces of the type G/H_i .

The group G was described in Subsection 1.6. Now we want to describe the stabilizer H of the pair of subspaces (Q_1, Q_2) . For this purpose we define the form $D'(v, w) = B(v, Jw)$ on subspace Q_1 (it is the restriction of managing form D to the subspace Q_1). It is easy to see that $H \simeq \mathcal{U}(D')$.

Below we give a list of symmetric spaces obtained in this way.

The first line indicate the symmetric space G/H .

The second line contains the space $V = \mathbb{K}^l$ and type of the form B . In this line we also indicate type of the semiinvolution J

The third line contains the group $\mathcal{U}(B)$. For us it is more pleasant to denote it by G^* (in some cases this information also give precise inertia indexes of the form B).

The forth line contains the centralizer GL^J of the semiinvolution J in $\mathrm{GL}(V)$ and also the group $\mathcal{U}(D)$ consisting of operators preserving managing form $D(v, w) = B(v, Jw)$. Call to mind that equalities (1.3)–(1.5) are fulfilled.

We mark by the symbol \star the cases when $\mathcal{S}(B, J)$ is not G -homogeneous space. In this case we give a decomposition of $\mathcal{S}(B, J)$ onto an union of symmetric spaces.

1. $O(p, q) \times O(p, q)/O(p, q)$
 $V = \mathbb{R}^{2(p+q)}, B$ is symmetric, $J^2 = 1, \quad B(Jv, Jw) = B(v, w).$
 $G^* = O(p+q, p+q),$
 $GL^J = GL(p+q, \mathbb{R}) \times GL(p+q, \mathbb{R}) \quad \mathcal{U}(D) = O(2p, 2q).$
In particular $O(p) \times O(p)/O(p)$ is a compact symmetric space
2. $Sp(2n, \mathbb{R}) \times Sp(2n, \mathbb{R})/Sp(2n, \mathbb{R})$
 $V = \mathbb{R}^{4n}, \quad B$ is skewsym., $J^2 = 1, \quad B(Jv, Jw) = B(v, w).$
 $G^* = Sp(4n, \mathbb{R})$
 $GL^J = GL(2n, \mathbb{R}) \times GL(2n, \mathbb{R}), \quad \mathcal{U}(D) = Sp(4n, \mathbb{R})$
3. $GL(n, \mathbb{R})/O(p, n-p)$
 $V = \mathbb{R}^{2n}, \quad B$ is skewsym., $J^2 = 1, \quad B(Jv, Jw) = -B(v, w).$
 $G^* = Sp(2n, \mathbb{R})$
 $GL^J = GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \quad \mathcal{U}(D) = O(n, n)$
 $\mathcal{S}(B, J) = \cup_{p=0}^n GL(n, \mathbb{R})/O(p, n-p)$
In particular $GL(n, \mathbb{R})/O(n)$ is a noncompact symmetric space
4. $GL(2n, \mathbb{R})/Sp(2n, \mathbb{R})$
 $V = \mathbb{R}^{4n}, \quad B$ is symmetric, $J^2 = 1, \quad B(Jv, Jw) = -B(v, w).$
 $G^* = O(2n, 2n)$
 $GL^J = GL(2n, \mathbb{R}) \times GL(2n, \mathbb{R}), \quad \mathcal{U}(D) = Sp(2n, \mathbb{R})$
5. $O(n, \mathbb{C})/O(p, n-p)$
 $V = \mathbb{R}^{2n}, \quad B$ is symmetric, $J^2 = -1, \quad B(Jv, Jw) = -B(v, w).$
 $G^* = O(n, n)$
 $GL^J = GL(n, \mathbb{C}), \quad \mathcal{U}(D) = O(n, n)$
 $\mathcal{S}(B, J) = \cup_{p=0}^n O(n, \mathbb{C})/O(p, n-p)$
In particular $O(n, \mathbb{C})/O(n)$ is a noncompact symmetric space
6. $Sp(2n, \mathbb{C})/Sp(2n, \mathbb{R})$
 $V = \mathbb{R}^{4n}, \quad B$ is skewsym., $J^2 = -1, \quad B(Jv, Jw) = -B(v, w).$
 $G^* = Sp(4n, \mathbb{R})$
 $GL^J = GL(2n, \mathbb{C}) \quad \mathcal{U}(D) = Sp(4n, \mathbb{R})$
7. $U(n, n)/Sp(2n, \mathbb{R})$
 $V = \mathbb{R}^{4n}, \quad B$ is symmetric, $J^2 = -1, \quad B(Jv, Jw) = B(v, w).$
 $G^* = O(2n, 2n)$
 $GL^J = GL(2n, \mathbb{C}) \quad \mathcal{U}(D) = Sp(4n, \mathbb{R})$
8. $U(p, q)/O(p, q)$
 $V = \mathbb{R}^{2(p+q)}, B$ is skewsym., $J^2 = -1, \quad B(Jv, Jw) = B(v, w).$
 $G^* = Sp(2(p+q), \mathbb{R})$
 $GL^J = GL(p+q, \mathbb{C}) \quad \mathcal{U}(D) = O(2p, 2q)$
In particular $U(p)/O(p)$ is a compact symmetric space

9. $O(n, \mathbb{C}) \times O(n, \mathbb{C})/O(n, \mathbb{C})$
 $V = \mathbb{C}^{2n}$, B is symmetric, J is linear, $J^2 = 1$, $B(Jv, Jw) = B(v, w)$.
 $G^* = O(2n, \mathbb{C})$,
 $GL^J = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ $\mathcal{U}(D) = O(2n, \mathbb{C})$
10. $Sp(2n, \mathbb{C}) \times Sp(2n, \mathbb{C})/Sp(2n, \mathbb{C})$
 $V = \mathbb{C}^{4n}$, B is skewsym., J is linear, $J^2 = 1$, $B(Jv, Jw) = B(v, w)$.
 $G^* = Sp(4n, \mathbb{C})$
 $GL^J = GL(2n, \mathbb{C}) \times GL(2n, \mathbb{C})$ $\mathcal{U}(D) = Sp(4n, \mathbb{C})$
11. $U(p, q) \times U(p, q)/U(p, q)$
 $V = \mathbb{C}^{2(p+q)}$, B is hermitian, J is linear, $J^2 = 1$, $B(Jv, Jw) = B(v, w)$.
 $G^* = U(p+q, p+q)$
 $GL^J = GL(p+q, \mathbb{C}) \times GL(p+q, \mathbb{C})$ $\mathcal{U}(D) = U(2p, 2q)$
In particular $U(p) \times U(p)/U(p)$ is a compact symmetric space
12. $GL(2n, \mathbb{C})/Sp(2n, \mathbb{C})$
 $V = \mathbb{C}^{4n}$, B is symmetric, J is linear, $J^2 = 1$, $B(Jv, Jw) = -B(v, w)$.
 $G^* = O(4n, \mathbb{C})$,
 $GL^J = GL(2n, \mathbb{C}) \times GL(2n, \mathbb{C})$ $\mathcal{U}(D) = Sp(4n, \mathbb{C})$
13. $GL(n, \mathbb{C})/O(n, \mathbb{C})$
 $V = \mathbb{C}^{2n}$, B is skewsym., J is linear, $J^2 = 1$, $B(Jv, Jw) = -B(v, w)$.
 $G^* = Sp(2n, \mathbb{C})$
 $GL^J = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ $\mathcal{U}(D) = O(2n, \mathbb{C})$.
- 14.* $GL(n, \mathbb{C})/U(p, n-p)$
 $V = \mathbb{C}^{2n}$, B is hermitian, J is linear, $J^2 = 1$, $B(Jv, Jw) = -B(v, w)$.
 $G^* = U(n, n)$
 $GL^J = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$, $\mathcal{U}(D) = U(n, n)$
 $\mathcal{S}(B, J) = \cup_{p=0}^n GL(n, \mathbb{C})/U(p, n-p)$
In particular $GL(n, \mathbb{C})/U(n)$ is a noncompact symmetric space
- 15.* $Sp(2n, \mathbb{R})/U(p, n-p)$
 $V = \mathbb{C}^{2n}$, B is skewsym., J is antilin., $J^2 = 1$, $B(Jv, Jw) = \overline{B(v, w)}$.
 $G^* = Sp(2n, \mathbb{C})$,
 $GL^J = GL(2n, \mathbb{R})$ $\mathcal{U}(D) = U(n, n)$
 $\mathcal{S}(B, J) = \cup_{p=0}^n Sp(2n, \mathbb{R})/U(p, n-p)$
In particular $Sp(2n, \mathbb{R})/U(n)$ is a noncompact symmetric space
16. $O(2p, 2q)/U(p, q)$
 $V = \mathbb{C}^{2(p+q)}$, B is symmetric, J is antilin., $J^2 = 1$, $B(Jv, Jw) = \overline{B(v, w)}$.
 $G^* = O(2(p+q), \mathbb{C})$
 $GL^J = GL(2(p+q), \mathbb{R})$ $\mathcal{U}(D) = U(2p, 2q)$
In particular $O(2p)/U(p)$ is a compact symmetric space

17. $O(n, n)/O(n, \mathbb{C})$
 $V = \mathbb{C}^{2n}$, B is hermitian, J is antilin., $J^2 = 1$, $B(Jv, Jw) = \overline{B(v, w)}$.
 $G^* = U(n, n)$,
 $GL^J = GL(2n, \mathbb{R})$ $\mathcal{U}(D) = O(2n, \mathbb{C})$
18. $Sp(4n, \mathbb{R})/Sp(2n, \mathbb{C})$
 $V = \mathbb{C}^{4n}$, B is hermitian, J is antilin., $J^2 = 1$, $B(Jv, Jw) = -\overline{B(v, w)}$.
 $G^* = U(2n, 2n)$,
 $GL^J = GL(4n, \mathbb{R})$, $\mathcal{U}(D) = Sp(4n, \mathbb{C})$
19. $Sp(p, q)/U(p, q)$
 $V = \mathbb{C}^{2(p+q)}$, B is skewsym., J is antilin., $J^2 = -1$, $B(Jv, Jw) = \overline{B(v, w)}$.
 $G^* = Sp(2(p+q), \mathbb{C})$
 $GL^J = GL(p+q, \mathbb{H})$, $\mathcal{U}(D) = U(2p, 2q)$
In particular $Sp(p)/U(p)$ is a compact symmetric space
- 20.* $SO^*(2n)/U(p, n-p)$
 $V = \mathbb{C}^{2n}$, B is symmetric, J is antilin., $J^2 = -1$, $B(Jv, Jw) = \overline{B(v, w)}$.
 $G^* = O(2n, \mathbb{C})$,
 $GL^J = GL(n, \mathbb{H})$ $\mathcal{U}(D) = U(n, n)$
 $S(B, J) = \cup_{p=1}^n SO^*(2n)/U(p, n-p)$
In particular $SO^*(2n)/U(n)$ is a noncompact symmetric space
21. $Sp(n, n)/Sp(2n, \mathbb{C})$
 $V = \mathbb{C}^{4n}$, B is hermitian, J is antilin., $J^2 = -1$, $B(Jv, Jw) = \overline{B(v, w)}$.
 $G^* = U(2n, 2n)$
 $GL^J = GL(2n, \mathbb{H})$ $\mathcal{U}(D) = Sp(4n, \mathbb{C})$
22. $SO^*(2n)/O(n, \mathbb{C})$
 $V = \mathbb{C}^{2n}$, B is hermitian, J is antilin., $J^2 = -1$, $B(Jv, Jw) = -\overline{B(v, w)}$.
 $G^* = U(n, n)$
 $G = GL(n, \mathbb{H})$, $\mathcal{U}(D) = O(2n, \mathbb{C})$
23. $Sp(p, q) \times Sp(p, q)/Sp(p, q)$
 $V = \mathbb{H}^{2(p+q)}$, B is hermitian, $J^2 = 1$, $B(Jv, Jw) = B(v, w)$.
 $G^* = Sp(p+q, p+q)$,
 $GL^J = GL(p+q, \mathbb{H}) \times GL(p+q, \mathbb{H})$ $\mathcal{U}(D) = Sp(2p, 2q)$
In particular $Sp(p) \times Sp(p)/Sp(p)$ is a compact symmetric space
24. $SO^*(2n) \times SO^*(2n)/SO^*(2n)$
 $V = \mathbb{H}^{2n}$, B is antihermitian, $J^2 = 1$, $B(Jv, Jw) = B(v, w)$.
 $G^* = SO^*(4n)$
 $GL^J = GL(n, \mathbb{H}) \times GL(n, \mathbb{H})$ $\mathcal{U}(D) = SO^*(4n)$

25. $\mathrm{GL}(n, \mathbb{H})/\mathrm{SO}^*(2n)$
 $V = \mathbb{H}^{2n}$, B is hermitian, $J^2 = 1$, $B(Jv, Jw) = -B(v, w)$.
 $G^* = \mathrm{Sp}(n, n)$,
 $\mathrm{GL}^J = \mathrm{GL}(n, \mathbb{H}) \times \mathrm{GL}(n, \mathbb{H})$, $\mathcal{U}(D) = \mathrm{SO}^*(4n)$
- 26.* $\mathrm{GL}(n, \mathbb{H})/\mathrm{Sp}(p, n-p)$
 $V = \mathbb{H}^{2n}$, B is antihermitian, $J^2 = 1$, $B(Jv, Jw) = -B(v, w)$.
 $G^* = \mathrm{SO}^*(4n)$
 $\mathrm{GL}^J = \mathrm{GL}(n, \mathbb{H}) \times \mathrm{GL}(n, \mathbb{H})$ $\mathcal{U}(D) = \mathrm{Sp}(n, n)$
 $\mathcal{S}(B, J) = \cup_{p=0}^n \mathrm{GL}(n, \mathbb{H})/\mathrm{Sp}(p, n-p)$
In particular $\mathrm{GL}(n, \mathbb{H})/\mathrm{Sp}(n)$ is a noncompact symmetric space
27. $\mathrm{U}(2p, 2q)/\mathrm{Sp}(p, q)$
 $V = \mathbb{H}^{2(p+q)}$, B is antihermitian, $J^2 = -1$, $B(Jv, Jw) = B(v, w)$.
 $G^* = \mathrm{SO}^*(4(p+q))$,
 $\mathrm{GL}^J = \mathrm{GL}(2(p+q), \mathbb{C})$, $\mathcal{U}(D) = \mathrm{Sp}(2p, 2q)$
In particular $\mathrm{U}(2p)/\mathrm{Sp}(p)$ is a compact symmetric space
28. $\mathrm{U}(n, n)/\mathrm{SO}^*(2n)$
 $V = \mathbb{H}^{2n}$, B is hermitian, $J^2 = -1$, $B(Jv, Jw) = B(v, w)$.
 $G^* = \mathrm{Sp}(n, n)$,
 $\mathrm{GL}^J = \mathrm{GL}(2n, \mathbb{C})$, $\mathcal{U}(D) = \mathrm{SO}^*(4n)$
29. $\mathrm{O}(2n, \mathbb{C})/\mathrm{SO}^*(2n)$
 $V = \mathbb{H}^{2n}$, B is antihermitian, $J^2 = -1$, $B(Jv, Jw) = -B(v, w)$.
 $G^* = \mathrm{SO}^*(4n)$
 $\mathrm{GL}^J = \mathrm{GL}(2n, \mathbb{C})$ $\mathcal{U}(D) = \mathrm{SO}^*(4n)$
- 30.* $\mathrm{Sp}(2n, \mathbb{C})/\mathrm{Sp}(p, n-p)$
 $V = \mathbb{H}^{2n}$, B is hermitian, $J^2 = -1$, $B(Jv, Jw) = -B(v, w)$.
 $G^* = \mathrm{Sp}(n, n)$
 $\mathrm{GL}^J = \mathrm{GL}(2n, \mathbb{C})$ $\mathcal{U}(D) = \mathrm{Sp}(n, n)$
 $\mathcal{S}(B, J) = \cup_{p=0}^n \mathrm{Sp}(2n, \mathbb{C})/\mathrm{Sp}(p, n-p)$
In particular $\mathrm{Sp}(2n, \mathbb{C})/\mathrm{U}(n)$ is a noncompact symmetric space

2.2. List 2. The case when we have only underlying form B . Let us fix a linear space $V = \mathbb{K}^{2n}$ equipped with a split form B . Define the space $\mathcal{S}(B)$. Points of $\mathcal{S}(B)$ are ordered pairs (Q_1, Q_2) of maximal isotropic subspaces in the V such that $V = Q_1 \oplus Q_2$. Obviously all spaces $\mathcal{S}(B)$ are symmetric spaces having the type

$$G/H = \mathcal{U}(B)/\mathrm{GL}(n, \mathbb{K})$$

Below we give the list of symmetric spaces obtained in this way

31. $\mathrm{O}(n, n)/\mathrm{GL}(n, \mathbb{R})$
32. $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{R})$

- 33. $O(2n, \mathbb{C})/\mathrm{GL}(n, \mathbb{C})$
- 34. $\mathrm{Sp}(2n, \mathbb{C})/\mathrm{GL}(n, \mathbb{C})$
- 35. $\mathrm{U}(n, n)/\mathrm{GL}(n, \mathbb{C})$
- 36. $\mathrm{Sp}(n, n)/\mathrm{GL}(n, \mathbb{H})$
- 37. $\mathrm{SO}^*(4n)/\mathrm{GL}(n, \mathbb{H})$

In all cases we define the group

$$G^* = G \times G$$

2.3. List 3. The case when we have only a managing semiinvolution. Consider a linear space $V = \mathbb{K}^{2n}$ and a split (see 1.2) semiinvolution J in V . Let us define the space $\mathcal{S}(J)$. Points of $\mathcal{S}(J)$ are all pairs of subspaces (Q_1, Q_2) in V such that $V = Q_1 \oplus Q_2$, $JQ_1 = Q_2$. The group $G := \mathrm{GL}^J$ (centralizer of the semiinvolution J) acts on $\mathcal{S}(J)$. It is readily seen that in all cases $\mathcal{S}(J)$ is a symmetric space.

The list of such spaces is given below. The first row indicates the space G/H . The second row contains the space V and the type of the semiinvolution J .

- 38. $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{R})$
 $V = \mathbb{R}^{2n}, J^2 = 1$
- 39. $\mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{R})$
 $V = \mathbb{R}^{2n}, J^2 = -1$
- 40. $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{C})$
 $V = \mathbb{C}^{2n}, J^2 = 1, J$
- 41. $\mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$
 $V = \mathbb{C}^{2n}, J^2 = 1, J$
- 42. $\mathrm{GL}(n, \mathbb{H})/\mathrm{GL}(n, \mathbb{C})$
 $V = \mathbb{C}^{2n}, J^2 = -1, J$
- 43. $\mathrm{GL}(n, \mathbb{H}) \times \mathrm{GL}(n, \mathbb{H})/\mathrm{GL}(n, \mathbb{H})$
 $V = \mathbb{H}^{2n}, J^2 = 1$
- 44. $\mathrm{GL}(2n, \mathbb{C})/\mathrm{GL}(n, \mathbb{H})$
 $V = \mathbb{H}^{2n}, J^2 = -1$

$$G^* = \mathrm{GL}(V)$$

2.4. List 4. The case when we have only managing form. Consider a space $V = \mathbb{K}^n$, equipped with a form D . Assume $G = \mathcal{U}(D)$. Consider the space $\mathcal{S}_m(D)$ consisting of all m -dimensional subspaces $Q_1 \subset V$ such that the form D is nondegenerate on Q_1 . We also can say that a point of the space $\mathcal{S}_m(D)$ is a pair of subspaces (Q_1, Q_2) such that

1. $V = Q_1 \oplus Q_2$,
2. Q_2 is D -orthogonal complement to Q_1 .

Obviously, either $\mathcal{S}_m(D)$ is a symmetric space of the type G/H or $\mathcal{S}_m(D)$ is an union of a finite family of symmetric spaces G/H_i . The list of such symmetric spaces is given below. In the case when the form D is hermitian we also describe decomposition of $\mathcal{S}_m(D)$ onto the union of symmetric spaces.

45.* $O(p, q)/O(r, s) \times O(p - r, q - s)$

$$\mathcal{S}_m(D) = \bigcup_{r, s: r+s=m, r \leq p, s \leq q} O(p, q)/O(r, s) \times O(p - r, q - s)$$

In particular $O(p, q)/O(p) \times O(q)$ is a noncompact symmetric space
 $O(p)/O(r) \times O(p - r)$ is a compact symmetric space

46. $Sp(2(k + l), \mathbb{R})/Sp(2k, \mathbb{R}) \times Sp(2l, \mathbb{R})$

47. $O(n + m, \mathbb{C})/O(n, \mathbb{C}) \times O(m, \mathbb{C})$

48. $Sp(2(k + l), \mathbb{C})/Sp(2k, \mathbb{C}) \times Sp(2l, \mathbb{C})$

49.* $U(p, q)/U(r, s) \times U(p - r, q - s)$

$$\mathcal{S}_m(D) = \bigcup_{r, s: r+s=m, r \leq p, s \leq q} U(p, q)/U(r, s) \times U(p - r, q - s)$$

In particular $U(p, q)/U(p) \times U(q)$ is a noncompact symmetric space
 $U(p)/U(r) \times U(p - r)$ is a compact symmetric space

50. $Sp(p, q)/Sp(r, s) \times Sp(p - r, q - s)$

$$\mathcal{S}_m(D) = \bigcup_{r, s: r+s=m, r \leq p, s \leq q} Sp(p, q)/Sp(r, s) \times Sp(p - r, q - s)$$

In particular $Sp(p, q)/Sp(p) \times Sp(q)$ is a noncompact symmetric space
 $Sp(p)/Sp(r) \times Sp(p - r)$ is a compact symmetric space

51. $SO^*(2(m + n))/SO^*(2m) \times SO^*(2n)$

In all cases we define the group

$$G^* = GL(V)$$

2.5. List 5. The case when there is nothing. Consider the space $V = \mathbb{K}^{p+q}$. Further consider the space \mathcal{S}_p , consisting of all pairs of subspaces (Q_1, Q_2) in V such that

1. $\dim Q_1 = p, \dim Q_2 = q$
2. $V = Q_1 \oplus Q_2$

By this way we obtain the following symmetric spaces \mathcal{S}

52. $GL(p + q, \mathbb{R})/GL(p, \mathbb{R}) \times GL(q, \mathbb{R})$

53. $GL(p + q, \mathbb{C})/GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$

54. $GL(p + q, \mathbb{H})/GL(p, \mathbb{H}) \times GL(q, \mathbb{H})$

In all cases we define the group

$$G^* = GL(p + q, \mathbb{K})$$

§3. Some applications

3.1. Open embeddings to grassmannians. Thus in all 54 cases a point of a symmetric space G/H is a pair of subspaces (Q_1, Q_2) in a linear space.

If we have managing semiinvolution or managing form (Lists 1,3,4), then the subspace Q_2 is uniquely defined by the subspace Q_1 . Hence the map $(Q_1, Q_2) \mapsto Q_1$ is an open embedding of the symmetric space $\mathcal{S} = G/H$ to some grassmannian Gr^* (this grassmannian is complete grassmannian for the Lists 3,4 and isotropic grassmannian $\text{Gr}(D)$ for the List 1).

Let we have no managing semiinvolution and no managing form (Lists 2,5). Then (Q_1, Q_2) is a point of products of two grassmannians. We also denote this product of grassmannians by Gr^* .

The image of the space G/H in the grassmannian (or product of two grassmannians) Gr^* in all cases is open. Moreover the image is open in all cases except 10 series marked by the symbol \star .

REMARK. If a space G/H is compact then its image coincides with grassmannian. In other words we realized all 10 series of compact symmetric spaces as grassmannians.

3.2. Overgroup. For all symmetric spaces G/H we indicated the group $G^* \supset G$. By the construction the group G^* acts transitively on the grassmannian Gr^* , A stabilizer of a point is a maximal parabolic subgroup in G^* .

3.3. Restriction from degenerated principal series. Consider the natural unitary representation ρ of the group G^* in the space L^2 on Gr^* .

PROPOSITION 3.1. a) *For all classical symmetric spaces except the cases G/H marked by \star the restriction of the representation ρ to the subgroup G is equivalent to the representation of G in $L^2(G/H)$.*

b) *For cases marked by \star the restriction of ρ to G is equivalent to the representation of G in $\bigoplus L^2(G/H_i)$ (where the spaces G/H_i are indicated in List.*

PROOF. It is an obvious consequence from Subsection 3.1. ■

REMARK. Consider the case $G^* = G \times G$ (Lists 2,5). Consider the representation of the group $G^* = G \times G$ in the space L^2 on the product of two grassmannians. Obviously this representation is a tensor product of two representations of the group G . Hence in this cases the representation of G in $L^2(G/H)$ is a tensor product of two representations of the group G of degenerated principal series.

3.4. Matrix coordinates on symmetric spaces. Consider a linear space V and a pair of subspaces X, Y such that $Z = X \oplus Y$. Let $\dim X = \alpha$. As before we denote by $\text{Gr}_\alpha(V)$ the grassmannian of all α -dimensional subspaces. Assume $R \in \text{Gr}_\alpha(Z)$ doesn't intersect with Y . Then R is a graph of some operator $X \rightarrow Y$. This operator is named *an angular operator* of subspace R , associated with the decomposition $V = X \oplus Y$.

Fix a decomposition $V = X \oplus Y$. We write elements of the group $\text{GL}(V)$ as block operators

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \oplus Y \rightarrow X \oplus Y$$

The action of the group $GL(V)$ on grassmannian on language of angular operators is given by the formula

$$R \rightarrow (C + DR)(A + BR)^{-1}$$

Let us consider a symmetric space G/H and let us realize it as a set \mathcal{S} . Call to mind that a point of \mathcal{S} is a ordered pair of subspaces (Q, R) . Fix a pair $(X, Y) \in \mathcal{S}$. For each point $(Q, R) \in \mathcal{S}$ we associate the pair of operators

$$(M, N)$$

where

$M : X \rightarrow Y$ is the angular operator of the subspace Q associated with the decomposition $X \oplus Y$ and

$N : Y \rightarrow X$ is the angular operator of the subspace R associated with the decomposition $Y \oplus X$.

The condition $Q \cap R = 0$ in our coordinates means

$$\det(1 - MN) \neq 0 \quad (3.1)$$

All other conditions also can be written in very simple form

a) Consider the case then Q, R (and in particular X, Y) are isotropic with respect to a form B (Lists 1,2). For for all $x_1, x_2 \in X$ we have $x_1 + Mx_1, x_2 + Mx_2 \in Q$. Hence

$$\begin{aligned} 0 &= B(x_1 + Mx_1, x_2 + Mx_2) = \\ &= B(x_1, x_2) + B(Mx_1, Mx_2) + B(Mx_1, x_2) + B(x_1, Mx_2) = \\ &= 0 + 0 + B(Mx_1, x_2) + B(x_1, Mx_2) \end{aligned}$$

(and similarly for N). Thus

$$B(Mx_1, x_2) + B(x_1, Mx_2) = 0 \quad (3.2)$$

In matrix coordinates it means that a matrix is symmetric, skewsymmetric, hermitian, antihermitian. (depending on a type of the form B)

b) Consider the case when R, Q are orthogonal with respect to a managing form D (Lists 1,4). Then for all $x \in Q, y \in R$

$$\begin{aligned} 0 &= D(x + Mx, y + Ny) = D(x, y) + D(Mx, Ny) + D(x, Ny) + D(Mx, y) = \\ &= 0 + 0 + D(x, Ny) + D(Mx, y) \end{aligned}$$

Thus

$$D(x, Ny) + D(Mx, y) = 0 \quad (3.3)$$

In matrix language it give condition $N = \pm M^*$ or $N = \pm M^t$ (depending on the type of the form D).

c) Consider the cases then R and Q are linked by managing semiinvolution J (Lists 1,3). Then we obtain the condition

$$N = JMJ^{-1} \quad (3.4)$$

d) In the cases marked by \star different symmetric spaces G/H_i are separated by the hypersurface $\det(1 - MN) = 0$.

REMARK. Emphasis that equations (3.2)–(3.4) are linear. For each point we $(X, Y) \in \mathcal{S}$ we constructed a map on the manifold $G/H = \mathcal{S}$. Thus we obtained atlas on the manifold G/H

REMARK. For riemann noncompact symmetric spaces our construction is equivalent to realization of the type "matrix ball" (see for instance [9], Addendum A).

3.5. Hua Loo Keng double ratio. Let $(Q_1, Q_2), (R_1, R_2)$ be points of a symmetric space $\mathcal{S} = G/H$. Let $M : Q_1 \rightarrow Q_2$ be the angular operator of the subspace R_1 associated to the decomposition $V = Q_1 \oplus Q_2$. Let $N : Q_2 \rightarrow Q_1$ be an angular operator of the subspace R_2 associated to the decomposition $V = Q_2 \oplus Q_1$. Then NM is a canonically defined operator $Q_1 \rightarrow Q_1$. Its eigenvalues $(\lambda_1, \lambda_1, \dots)$ are invariants of a pair of points $(Q_1, Q_2), (R_1, R_2)$ under the action of the group G . This construction is close to the usual double ratio of 4 points of projective line. For several series of classical symmetric spaces it was defined by Hua Loo Keng [7].

REMARK. Let (X_1, X_2) be coordinates of a point (Q_1, Q_2) , and (Y_1, Y_2) be coordinates of a point (R_1, R_2) in the sense of previous Subsection. Then double ratio coincides with eigenvalues of the matrix

$$(1 - Y_2 X_1)^{-1} (X_2 - Y_2) (1 - Y_1 X_2)^{-1} (X_1 - Y_1)$$

3.6. Goncharov–Gindikin conformal structures. Let Gr^* be one of our grassmannians. Fix a point $P \in \text{Gr}^*$ and an integer $\alpha = 1, 2, \dots, \dim P$. By $D_\alpha(P)$ we denote the space of all subspaces $Q \in \text{Gr}^*$ such that codimension of $P \cap Q$ in P is less or equals α . By T_P we denote the tangent space to Gr^* in a point P . Denote by $C_\alpha(P)$ the cone T_P consisting of vectors tangent to variety $D_\alpha(P)$. In this way for each α we obtaine a field of cones C_α on grassmannian Gr^* . Obviously this field of cones is G^* -invariant.

A.B.Goncharov and C.G.Gindikin (see [6]) considered the field C_1 (or C_2 if C_1 is empty). It appeared that this structure (*in the case or rank* > 1) "remember" the group G^* . precisely the pseudogroup of diffeomorphisms of G/H preserving field of cones C_1 is the group G^* (up to connected components).

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